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Small-Angle Approximation to the Transfer of Narrow Laser Beams in Anisotropic Scattering Media

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Small-Angle Approximation to the Transfer of Narrow Laser Beams in Anisotropic Scattering Media

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SUMMARY

The propagation of a laser beam in an optically dense medium such as a fog, dust storm, or smoke, is a problem of growing importance, both for communication and detection purposes. Although such dense media lead to a significant attenuation of the primary beam, much of the scattered radiation may still be found close to the beam axis and will, thus, be available for detection by a suitable detector. In this report we examine the spreading of a laser beam using the small-angle scattering approximation to the equation of transfer. This approximation, which assumes that most photons travel essentially parallel to the beam axis, has also been used to study the propagation of fast charged particles through metal foils. It appears to be equally suited to the study of the propagation of beams of visible or near infrared light through media such as fog, dust or smoke, where the scattering phase function is highly anisotropic.

This equation of transfer may be solved in closed form by the use of Fourier transform techniques. The resulting expressions are simplest for the radiance, or alternately the power received by a coaxial detector, rather than for the irradiance. In view of the assumptions involved in the small-angle approximation, it is the radiance which is of more interest anyway. The resulting expression involves a single integral from zero to infinity; the appendix outlines the procedure for its numerical evaluation.

In keeping with the approximate nature of our solution, and in order to fully exploit its mathematical simplicity, we have chosen simple analytic models for the forward peak of the scattering phase function, rather than using Mie theory. As well as the well-known Gaussian functional form, we also examine some exponential and binomial models. Our computational results indicate that, provided the parameters of the models are suitably selected, there is little to choose between the models, with the exception of the sea-water model, which we do not recommend.

Despite the relative simplicity of the expressions that we obtain, a number of authors have resorted to further approximations, in order to extract even simpler results. The method of Dolin and Fante starts by separating the scattered and unscattered beams. In the case of a Gaussian phase function, this method leads to a single finite integral, which shows reasonable agreement with our results. The method of Arnush and Stotts is essentially a low-frequency approximation, which yields reasonable results well away from the beam axis but leads to unphysical results close to the axis. The method of Tam and Zardecki involves a series expansion of our integral leading to a series of multidimensional finite integrals; it is applicable to both radiance and irradiance (which is its main advantage), though only for the Gaussian phase function.

1.0 INTRODUCTION

If a relatively narrow beam propagates in a scattering medium, photons are constantly removed from the beam. However, if the scatterers are of a size equal to or greater than the radiation wavelength, such as in the case of smoke, dust or fog particles compared to visible wavelengths, then most scattering events will result in a comparatively small deflection of the photon. This may lead to a gradual spreading of the original beam, both in thickness and angle.

In this report we examine the broadening of a laser beam, and the signal that may be detected, as functions of both experimental geometry and the properties of the scattering medium. We shall employ the small-angle approximation to the equation of radiative transfer, which ignores photons which have suffered large deflections as they will be assumed lost. In order to obtain tractable answers, it will prove necessary to assume simple analytic forms for the scattering phase function and the initial beam profile. Nevertheless, the analysis presented in this report will be as free as possible of unnecessary approximations.

2.0 EQUATION OF TRANSFER IN SMALL-ANGLE APPROXIMATION

Let $I(z, \underline{r}, \hat{n}) d\underline{v} d\hat{n}$ be the intensity of radiation (or the number of photons) in a volume element $d\underline{v}$ centered at the point $z, \underline{r} = (x, y)$, and travelling within a cone of solid angle $d\hat{n}$ centered about the direction \hat{n} . Then I satisfies the equation of radiative transfer, which we may write (Refs. 1 and 2):

$$\left(\hat{n} \cdot \nabla + \sigma \right) I(z, \underline{r}, \hat{n}) = \omega_0 \sigma \int P(\hat{n} \cdot \hat{n}') I(z, \underline{r}, \hat{n}') d\hat{n}' \quad (1)$$

where σ is the extinction coefficient (km^{-1}),

ω_0 is the albedo of single scattering

$\psi = \cos^{-1} (\hat{n} \cdot \hat{n}')$ is the scattering angle

and $P(\psi)$ is the scattering phase function.

Even allowing for cylindrical symmetry about the axis of propagation (the z axis), Eq. (1) is exceedingly hard to solve, even numerically. However, since the diameter of our detector will always be small compared to the total propagation distance, we may safely assume that all photons which are eventually detected will have spent their flight time travelling essentially parallel to the z -axis. We may thus set $\cos \theta = 1$, where θ is the angle the photon makes with the z axis.

Note that this approximation ignores the contribution from all photons which undergo at least one large-angle scattering event. All such photons will clearly need to undergo at least a second large-angle scattering event, and maybe even a third, in order for them to reach the detector. As we are assuming that the phase function, P , is strongly forward-peaked, the probability of two or more large-angle scatterings is clearly very small, and thus the neglected contribution will be small.

The main effect of this assumption is to replace the unit propagation vector by

$$\hat{n} \equiv (n_z, n_{\perp}) \rightarrow (1, n_{\perp}) . \quad (2a)$$

Although this new propagation vector is no longer correctly normalized, this should not cause any problems, as the number of photons for which $|n_{\perp}| \ll 1$ is not true will clearly be small.

The second effect is that we may use $\underline{n}_\perp - \underline{n}'_\perp$ as the argument of P in Eq. (1), i.e.,

$$P(\underline{\hat{n}} \cdot \underline{\hat{n}}') \rightarrow P(\underline{n}_\perp - \underline{n}'_\perp). \quad (2b)$$

The third effect is to replace the limits of this (two-dimensional) integral by $\pm \infty$. With these points in mind, we may now rewrite Eq. (1) as

$$\left(\frac{\partial}{\partial z} + \underline{n}_\perp \cdot \frac{\partial}{\partial \underline{r}} + \sigma \right) I(z, \underline{r}, \underline{n}_\perp) = \omega_0 \sigma \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P(\underline{n}_\perp - \underline{n}'_\perp) I(z, \underline{r}, \underline{n}'_\perp) d\underline{n}'_\perp. \quad (3)$$

Equation (3) is referred to as the equation of radiative transfer in the small-angle approximation. Its main advantage over Eq. (1) is in the simplification of the directional derivative. This equation has been used extensively in the theory of foil penetration by fast charged particles (Refs. 3, 4 and 5). Though Wentzel (Ref. 3) was the first to use the small-angle approximation for charged particle transfer, perhaps the first person to employ this equation in the field of radiative transfer appears to be Dolin (Ref. 2).

One further result of the small-angle approximation is that all detected photons are assumed to have travelled the same distance. Thus their time of travel is constant, and a pulse will undergo no time-dispersion.

3.0 FORMAL SOLUTION IN THE SMALL-ANGLE APPROXIMATION

Equation (3) may be solved, at least formally, by the use of Fourier transform techniques. Introducing the definitions

$$\hat{I}(z, \underline{\eta}, \underline{\xi}) = (2\pi)^{-2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} I(z, \underline{r}, \underline{n}_1) e^{i(\underline{\eta} \cdot \underline{r} + \underline{\xi} \cdot \underline{n}_1)} d\underline{r} d\underline{n}_1 \quad (4)$$

$$\text{and } \hat{P}(\underline{\xi}) = (2\pi)^{-1} \int_{-\infty}^{\infty} P(\underline{n}_1) e^{i \underline{\xi} \cdot \underline{n}_1} d\underline{n}_1, \quad (5)$$

we take the double Fourier transform of Eq. (3) to obtain (Ref. 2)

$$\left(\frac{\partial}{\partial z} - \underline{\eta} \cdot \frac{\partial}{\partial \underline{\xi}} + \sigma \right) \hat{I}(z, \underline{\eta}, \underline{\xi}) = 2\pi \omega_0 \sigma \hat{P}(\underline{\xi}) \hat{I}(z, \underline{\eta}, \underline{\xi}). \quad (6)$$

Equation (6) is easily solved, to yield (Refs. 2, 5)

$$\hat{I}(z, \underline{\eta}, \underline{\xi}) = \hat{I}_0(\underline{\eta}, \underline{\xi} + z \underline{\eta}) e^{-\sigma z + \Omega} \quad (7)$$

where

$$\Omega \equiv \Omega(\underline{\xi}, \underline{\eta}) = 2\pi \omega_0 \sigma \int_0^z \hat{P}(|\underline{\xi} + z' \underline{\eta}|) dz' \quad (8)$$

and $\hat{I}_0(\underline{\eta}, \underline{\xi})$ is the Fourier transform of the initial intensity distribution (incident beam profile) at $z = 0$.

To obtain the intensity distribution at any point in the medium, it is merely necessary to re-transform Eq. (7) (Refs. 6 and 7)

$$I(z, \underline{r}, \underline{n}_1) = (2\pi)^{-2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{I}(z, \underline{\eta}, \underline{\xi}) e^{-i(\underline{\eta} \cdot \underline{r} + \underline{\xi} \cdot \underline{n}_1)} d\underline{\eta} d\underline{\xi}. \quad (9)$$

The principal difficulty with this procedure is the evaluation, and subsequent exponentiation, of the function Ω . Evaluation of Ω in the case of a real (Mie) phase function would appear to be prohibitive, and we shall employ instead a number of simpler analytic functions for P (described in later sections). However, even with such simplifying assumptions, many authors have still found it necessary to employ further approximations in order to obtain tractable expressions for \hat{I} (Eq. 7). We shall examine two of these approximations later.

Before we proceed to specific examples, however, we remind ourselves that it is irradiance (flux density) and received power, rather than radiance, which is of concern to us in this study. Using the relation between irradiance, N , and radiance, I , we may simplify Eq. (9) (Refs. 2 and 6).

$$\begin{aligned}
 N(z, \underline{r}) &\equiv \int_{2\pi} I(z, \underline{r}, \underline{n}_{\perp}) [\hat{n} \cdot \hat{z}] d\hat{n} \\
 &\approx \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} I(z, \underline{r}, \underline{n}_{\perp}) d\underline{n}_{\perp}
 \end{aligned} \tag{10}$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{I}(z, \underline{\eta}, 0) e^{-i \underline{\eta} \cdot \underline{r}} d\underline{\eta} \tag{11}$$

With the elimination of ξ in Eq. (11), we may simplify Eqs. (7) and (8):

$$\hat{I}(z, \underline{\eta}, 0) = \hat{I}_0(\underline{\eta}, z, \underline{\eta}) e^{-\sigma z + \Omega_0} \tag{7'}$$

$$\text{where } \Omega_0 = 2\pi \omega_0 \sigma \int_0^z \hat{P}(|z' - \eta|) dz' . \quad (8')$$

Equation (11) can now be further simplified by an appeal to symmetry. Since $P(\psi)$ clearly depends only on the scalar $|\underline{n}_\perp|$, and not on the vector \underline{n}_\perp , \hat{P} will similarly be a scalar function, as will Ω_0 . Similarly, if we assume that the incident beam profile is circularly symmetric, then $\hat{I}_0(\eta, z, \eta)$ will also be a scalar function of η . Thus $\hat{I}(z, \eta, 0)$ will be a scalar function, and Eq. (11) becomes

$$N(z, r) = 2\pi \int_0^\infty J_0(\eta r) \hat{I}(z, \eta, 0) \eta d\eta . \quad (11')$$

From Eq. (11') we see immediately that N is a scalar function of r , as we would expect from the above symmetry arguments. A more tractable expression for the case of a δ -function beam will be given in Eq. (31).

In most instances, of course, what we are most interested in (and what we physically measure) is the power received by some detector. This will involve the integration of Eq. (11') over the area of the detector, perhaps modulated by a response function. If we assume a coaxial, circular detector of radius R , with a flat response, then we have

$$\begin{aligned} P(z, R) &= 2\pi \int_0^R N(z, r) r dr \\ &= 4\pi^2 e^{-\sigma z} \int_0^\infty J_1(\eta R) \hat{I}_0(\eta, z, \eta) e^{\Omega_0 R} \eta d\eta . \end{aligned} \quad (12)$$

This result should prove amenable to numerical integration, especially if a relatively simple expression for $\hat{\Omega}_0$ can be obtained. Sample results will be presented below.

One useful result which can be obtained analytically, is the total power crossing a surface $z = \text{constant}$:

$$\begin{aligned}
 P(z, \infty) &= \int_{-\infty}^{\infty} N(z, r) dr \\
 &= \int \int \int \int \hat{I}(z, \eta, 0) e^{-i \eta \cdot r} d\eta dr \\
 &= 4\pi^2 \hat{I}(z, 0, 0) \\
 &= 4\pi^2 \hat{I}_0(0, 0) e^{-\sigma z + 2\pi \omega_0 \sigma z} \hat{P}(0) \\
 &= F_0 e^{-(1 - \omega_0) \sigma z}
 \end{aligned} \tag{13}$$

where F_0 is the incident total power, and we have used the fact that $\hat{P}(0) = (2\pi)^{-1}$.

From Eq. (13) we see that the only energy removed from the beam is that lost by absorption -- i.e., there is no backscatter.

One further parameter which will often prove useful is the beam spread, which we may define as

$$\begin{aligned}
 \langle r^2 \rangle &= \int_0^{\infty} N(z, r) r^3 dr / \int_0^{\infty} N(z, r) r dr \\
 &= 2\pi F_0^{-1} e^{(1 - \omega_0) \sigma z} \int_0^{\infty} N(z, r) r^3 dr .
 \end{aligned} \tag{14}$$

4.0 EXACT SOLUTION FOR GAUSSIAN BEAMS

At the entrance to a scattering medium, a laser beam profile can often be adequately represented by a Gaussian functional form, both for the radial distribution, and the angular divergence. Thus we have

$$I_0(r, \eta_{\perp}) = F_0 \beta^2 \gamma^2 \pi^{-2} \exp(-\beta^2 \eta_{\perp}^2 - \gamma^2 r^2). \quad (15)$$

This may be easily transformed, and, in particular, we have

$$\hat{I}_0(\eta, z, \eta) = F_0 (2\pi)^{-2} \exp(-\eta^2 / 4 \gamma^2 - z^2 \eta^2 / 4 \beta^2). \quad (16)$$

In general, the laser beam profile will be well collimated, so that β and γ will be large. The inclusion of Eq. (16) in Eq. (12) will in no way complicate the numerical integration, though in our examples later we will allow both to go to infinity, so as to reduce the number of parameters whose influence should be examined. In all practical calculations, however, realistic values of both parameters should be included.

4.1 Gaussian Phase Functions

A Gaussian functional form is also often employed to describe $P(\psi)$, since exact Mie theory is clearly somewhat impractical. Thus we choose to write

$$P(\psi) = 2 \alpha^2 e^{-\alpha^2 \psi^2} / 2\pi. \quad (17)$$

The α is an adjustable parameter, which controls the shape of the forward peak, and is related to the rms scattering angle $\overline{\psi^2}$ defined as

$$\overline{\psi^2} = 2\pi \int_0^\infty P(\psi) \psi^3 d\psi. \quad (17a)$$

It is easily shown that for the Gaussian case

$$\overline{\psi^2} = \alpha^{-2}. \quad (17b)$$

(Though α will usually be large, it will rarely, if ever, be as large as β or γ .) Taking the Fourier transform of Eq. (17), we find

$$\hat{P}(\xi) = \int_0^\infty J_0(\xi \psi) P(\psi) \psi d\psi \quad (5')$$

$$= e^{-\xi^2/4 \alpha^2} / 2\pi \quad (18)$$

$$\text{and hence } \Omega_0 = \omega_0 \sigma \eta^{-1} \alpha \sqrt{\pi} \operatorname{erf}(z\eta/2\alpha) \quad (19)$$

where erf is the well-known error function.

Substituting Eqs. (16) and (19) into (12) we find

$$P(z, R) = F_0 R \int_0^\infty J_1(\eta R) \exp \left[\omega_0 \sigma \eta^{-1} \alpha \sqrt{\pi} \operatorname{erf}(z\eta/2\alpha) - \sigma z - \eta^2/4 \gamma^2 - z^2 \eta^2/4 \beta^2 \right] d\eta. \quad (20)$$

Equation (20) is an exact equation, which is solved numerically for various values of β and γ . Here, we make a simplification in Eq. (20) and consider the limiting case of $\beta, \gamma \rightarrow \infty$. This physically implies that the beam is collimated and has zero width in space. Then by making the variable changes

$$\tau = \sigma z$$

$$\tau_s = \omega_0 \tau$$

$$G = R/z (\psi^2)^{1/2}$$

$$\chi = \eta R$$

(21)

Equation (20) can be reduced to

$$P(z, R) = F_0 e^{-\tau} \int_0^\infty J_1(\chi) \exp \left[\tau_s \sqrt{\pi} G \chi^{-1} \operatorname{erf}(\chi/2G) \right] d\chi. \quad (22)$$

The power of the unscattered beam at an optical depth of τ is, of course, $F_0 e^{-\tau}$. Thus, the presence of forward scattering has increased the detected power by the factor

$$A(\tau_s, G) = \int_0^\infty J_1(\chi) \exp \left[\tau_s \sqrt{\pi} G \chi^{-1} \operatorname{erf}(\chi/2G) \right] d\chi. \quad (23)$$

A , which we call the amplification factor, is a function of two parameters; τ_s , the scattering optical thickness, and G , the geometry factor.

... Finally, we may obtain the beam spread by substituting from Eqs. (11) and...

(16) into Eq. (14):

$$\langle r^2 \rangle = \tau_s z^2 / 3\alpha^2 + z^2 / \beta^2 + \gamma^{-2}. \quad (24)$$

In general, the first term should dominate, except perhaps close to the point of entry into the medium.

4.2 Non-Gaussian Phase Functions

Although the Gaussian form in Eq. (17) is a popular model for the forward peak of the phase function, it is often a good idea to examine other models, to make sure that none of the results are simply an artifact of the Gaussian model. In this section, therefore, we shall examine a number of other functional forms which may be (and have also been) used to model anisotropic phase functions. We shall follow essentially the same steps as in the previous section, and present only the results, unless further explanation is necessary.

4.2.1 Exponential Phase Functions

$$i) \quad P(\psi) = \alpha^2 e^{-\alpha \psi} / 2\pi \quad (25a)$$

$$\hat{P}(\xi) = \alpha^3 (\alpha^2 + \xi^2)^{-3/2} / 2\pi \quad (25b)$$

$$\Omega_0 = \tau_s (1 + y^2/6)^{-1/2} \quad (25c)$$

where

$$y = z \eta / \alpha = \chi / G. \quad (26)$$

Thus

$$A(\tau_s, G) = \int_0^\infty J_1(\chi) \exp\left[\tau_s (1 + \chi^2/6 G^2)^{-1/2}\right] d\chi \quad (25d)$$

and

$$\langle r^2 \rangle = 2 \tau_s z^2 / \alpha^2 + z^2 / \beta^2 + \gamma^{-2} \quad (25e)$$

$$ii) \quad P(\psi) = \alpha \psi^{-1} e^{-\alpha \psi} / 2\pi \quad (27a)$$

$$\hat{P}(\xi) = \alpha(\alpha^2 + \xi^2)^{-1/2} / 2\pi \quad (27b)$$

$$\Omega_0 = \sqrt{2} \tau_s y^{-1} \ln \left[y/\sqrt{2} + (1 + \frac{1}{2} y^2)^{1/2} \right] \quad (27c)$$

Hence

$$A(\tau_s, G) = \int_0^\infty J_1(\chi) \exp\left\{ \tau_s \sqrt{2} G \chi^{-1} \ln \left[\chi/G\sqrt{2} + (1 + \frac{1}{2} \chi^2 G^{-2})^{1/2} \right] \right\} d\chi \quad (27d)$$

$$\langle r^2 \rangle = 2 \tau_s z^2 / 3 \alpha^2 + z^2 / \beta^2 + \gamma^{-2}.$$

Note that, although Eq. (27a) implies $P(0) = \infty$, the inclusion of the correct solid angle factor leads to a finite result for the amount of light scattered through any angle. In fact, Eq. (27a) has been employed by Bravo-Zhivotovskiy et al. (Ref. 6) to model the phase function of sea water.

4.2.2 Binomial Phase Functions

This time, we consider phase functions based on the functional form $(1 + \alpha^2 \psi^2)^{-\mu-1}$. We will need the result that

$$\int_0^\infty J_0(\eta\psi) (1 + \alpha^2 \psi^2)^{-\mu-1} \psi d\psi = (\eta/2\alpha)^\mu K_\mu(\eta/\alpha) / \alpha^2 \Gamma(\mu+1) \quad (28)$$

where K_μ is the modified Bessel function of the second kind. Thus if

$$P_{\mu}(\psi) = 2 \mu \alpha^2 (1 + \alpha^2 \psi^2)^{-\mu-1} / 2\pi \dots \dots \dots (29a)$$

$$\hat{P}_{\mu}(\xi) = (\xi/2\alpha)^{\mu} K_{\mu}(\xi/\alpha) / \pi \Gamma(\mu) \quad (29b)$$

$$\Omega_0 = \tau_s \sqrt{\pi} \Gamma(\mu + \frac{1}{2}) [K_{\mu}(y') L_{\mu-1}(y') + K_{\mu-1}(y') L_{\mu}(y')] / \Gamma(\mu) \quad (29c)$$

where $y' = y\sqrt{\mu-1}$

and L_{μ} is the modified Struve function of order μ .

The expression for A may be easily written down. For $\mu > 1$, we may obtain the beam spread:

$$\langle r^2 \rangle = \tau_s z^2/3 \alpha^2 (\mu - 1) + z^2/\beta^2 + \gamma^{-2}. \quad (29d)$$

Note that if μ is an odd half-integer, Eq. (29c) may be expressed in terms of exponential functions. For example, for $\mu = 3/2$ we find

$$\Omega_0(3/2) = \tau_s [2 \sqrt{2} y^{-1} - e^{-y/\sqrt{2}} (1 + 2 \sqrt{2} y^{-1})] \quad (30)$$

$$A(\tau_s, G) = \int_0^{\infty} J_1(\chi) \exp \{ \tau_s [2 \sqrt{2} G \chi^{-1} - e^{-\chi/G\sqrt{2}} (1 + 2 \sqrt{2} G \chi^{-1})] \} d\chi. \quad (30')$$

4.3 Beam Profile

The spreading of a laser beam in a forward-scattering medium can be most simply described via the beam spread parameter, $\langle r^2 \rangle$, which we have derived above. However, the profile of the expanding beam is also of interest and will now be considered.

The formal expression for irradiance versus distance from the axis is given by Eq. (11'). In most cases, this expression is well-behaved. However, in the special case of $\beta, \gamma \rightarrow \infty$, Eq. (11') will diverge. An alternative expression for $N(r)$ may be obtained either by an integration by parts of Eq. (11') or by differentiating Eq. (12). Adopting the second approach and setting $\hat{I}_0 = F_0 (2\pi)^{-2}$, we obtain

$$N(r) = \frac{1}{2\pi R} \frac{dP}{dR}$$

$$= - (2\pi)^{-1} F_0 r^{-2} e^{-\tau} \int_0^\infty J_1(\chi) e^{\Omega_0} \Omega'_0 y d\chi \quad (31)$$

where the prime denotes differentiation of Ω_0 with respect to y (Eq. 26). With the exception of the sea-water phase function, Ω_0 goes as y^{-1} for large y , and so Ω'_0 goes as y^{-2} , and convergence is assured.

5.0 APPROXIMATE SOLUTIONS

In an effort to simplify the above analysis, several authors have employed a number of approximations. In this section we shall examine two of these approximations, neither of which appears to be particularly useful in our problem.

5.1 Dolin-Fante Method

Dolin (Ref. 8) and Fante (Ref. 9) have argued that the angular shape of the scattered intensity should be a much more slowly varying function than $P(\psi)$, and have thus extracted it from the integral on the right side of Eq. (3). After separating the scattered intensity from the unscattered, and Fourier transforming, they arrive at the following expressions

$$\hat{I}^u(z, \eta, 0) = e^{-\sigma z} \hat{I}_0(\eta, z, \eta) \quad (32a)$$

$$\begin{aligned} \text{and } \hat{I}^s(z, \eta, 0) = \hat{I}_0(\eta, z, \eta) \int_0^z dz' \exp \left[-\sigma z' - \int_{z'}^z \lambda(t, \eta, z, \eta) dt \right] \\ \cdot \omega_0 \sigma \hat{P}[\eta(z - z')] \end{aligned} \quad (32b)$$

$$\text{where } \lambda(t, \eta, z, \eta) = \frac{1}{4} \omega_0 \sigma \overline{\psi^2} |z - \eta - t\eta|^2 + (1 - \omega_0) \sigma \quad (33a)$$

$$\int_{z'}^z \lambda(t, \eta, z, \eta) dt = \frac{1}{12} \omega_0 \sigma \overline{\psi^2} \eta^2 (z - z')^3 + (1 - \omega_0) \sigma (z - z') \quad (33b)$$

and $\overline{\psi^2}$ is defined by Eq. (17a).

Equations (32) and (33) may now be inserted in Eqs. (11) or (12) as required.

Although \hat{P} is no longer exponentiated, this result is complicated by the additional (finite) integration over z' . In the case of a Gaussian phase function, it is possible to reverse the orders of these two integrals, and perform that over η , to give

$$A = e^{\omega_0 \tau} - \omega_0 \tau \int_0^1 dt \exp \left[\omega_0 \tau t - G^2 / \left(\frac{1}{3} \omega_0 \tau t^3 + t^2 \right) \right]. \quad (34)$$

Our calculations show that this approximation is reasonably accurate, except in those situations where τ is large and G is small. In section 7 (Numerical Results), we will compare the predictions of Eq (34) with those of Eq. (23). As it is not possible to perform any of the integrals for any of the other phase function models, we have limited our examination of this approximation to the case of the Gaussian phase function.

5.2 Arnush-Stotts Method

In order to extract analytic answers, Arnush (Ref. 10) and Stotts (Ref. 11, 12) have expanded \hat{P} to second order before performing the integration to obtain Ω . (Series expansion of Ω would yield the same result.) Arnush has used Bravo-Zhivotovskiy's (Ref. 6) sea water phase function, Eq. (27a), while Stotts originally used a Gaussian phase function, Eq. (17), and more recently the sea water phase function. This approximation is sufficient to provide the correct values for both $P(z, \infty)$, and $\langle r^2 \rangle$.

We start by re-writing the definition of Ω_0 as follows

$$\begin{aligned}\Omega_0 &= 2\pi \omega_0 \sigma z (\eta z)^{-1} \int_0^{\eta z} \hat{P}(t) dt \\ &= 2\pi \omega_0 \sigma z (\eta z)^{-1} \int_0^{\eta z} \int_0^\infty J_0(t\psi) P(\psi) \psi d\psi dt.\end{aligned}\quad (35)$$

Expanding J_0 as a power series leads to

$$\Omega_0 = \omega_0 \sigma z (1 - \eta^2 z^2 \overline{\psi^2}/12 + \dots) \quad (36)$$

where $\overline{\psi^2}$ is defined by Eq. (17a).

It is complicated, but reasonably straightforward to obtain the following expression for the beam spread parameter:

$$\langle r^2 \rangle = \frac{1}{3} \omega_0 \sigma z^2 \psi^2 + z^2 / \beta^2 + \gamma^{-2}. \quad (37)$$

Ignoring the higher order terms in Eq. (36), we may insert this expression into Eq. (11') to obtain

$$N(z, r) = F_0 \exp \left[- (1 - \omega_0) \tau - r^2 / \langle r^2 \rangle \right] / \pi \langle r^2 \rangle. \quad (38)$$

Similarly, integration of Eq. (12) leads to

$$P(z, R) = F_0 e^{-(1 - \omega_0) \tau} \left[1 - e^{-R^2 / \langle r^2 \rangle} \right]. \quad (39)$$

Ignoring β and γ , we may re-express Eq. (39) in more familiar terms

$$P(z, R) = F_0 e^{-(1 - \omega_0) \tau} \left[1 - \exp \left(- 3 G^2 / \tau_s \right) \right], \quad (39')$$

$$\text{i.e., } A(\tau_s, G) = e^{\tau_s} \left[1 - \exp \left(- 3 G^2 / \tau_s \right) \right]. \quad (40)$$

Expansion of Ω_0 to second order in y is equivalent to an asymptotic expansion to second order in G^{-1} , or R^{-1} . Thus we may expect this approximation to be accurate for large values of G or R . However, its behaviour for small values of these parameters is quite different from that of the exact results

quoted in previous sections. Thus we cannot expect this approximation to prove particularly useful for our problem, as shown later by numerical comparisons.

In fact, one finds values of A which are less than unity!

For our problem, of course, we are concerned with small values of R and G , and hence we are interested in the behaviour of Ω_0 for large values of y , i.e., its asymptotic expansion. The phase function model of Eq. (27a) does not have an asymptotic expansion, due to the fact that $P(0) = \infty$. For the other cases we may easily show that

$$\Omega_0 \sim \tau_s \tilde{q}/y \quad (41)$$

$$\text{where } \tilde{q} = 2\pi \int_0^\infty P(\psi) d\psi / \alpha. \quad (42)$$

Thus for a Gaussian (Eq. (17)), $\tilde{q} = \sqrt{\pi}$; Eq. (25a) gives $\tilde{q} = 1$; and Eq. (29a) gives $\tilde{q} = \mu B(\frac{1}{2}, \mu + \frac{1}{2})$, where B is the beta function. (For $\mu = \frac{3}{2}$, for example, $\tilde{q} = 2$.)

For large values of y , the Arnush-Stotts approximation to Ω_0 goes to (minus) infinity, and so we cannot expect this approximation to accurately predict the power received by a small detector.

6.0 EXACT METHOD OF TAM AND ZARDECKI

The method of Tam and Zardecki (Ref. 7) is exact, at least in principle, but requires the evaluation of multidimensional integrals, the order of which is equal to the order of multiple scattering involved. We will restrict this discussion to the case of the Gaussian phase function only.

The Tam and Zardecki method consists in expanding $\exp(\Omega_0)$ in a Taylor series, before performing the integration over z' (Eq. 8'). Thus, inserting Eq. (18) in Eq. (8'), and performing the Taylor expansion, yields

$$\exp(\Omega_0) = 1 + \sum_{m=1}^{\infty} \frac{\tau_s^m}{z^m m!} \int_0^z \cdots \int_0^z dz_1 \cdots dz_m \exp \left\{ -\eta^2 \sum_{i=1}^m z_i^2 / 4\alpha^2 \right\}. \quad (43)$$

We may now perform the inverse Fourier transform (Eq. 11') to yield

$$N(z, r) = F_0 e^{-\tau} (\pi z^2)^{-1} \sum_{m=0}^{\infty} \frac{\tau_s^m}{m!} N_m(z, r) \quad (44)$$

$$\text{where } N_0 = \frac{z^2 \beta^2 \gamma^2}{z^2 \gamma^2 + \beta^2} \exp \left\{ -\frac{r^2 \beta^2 \gamma^2}{z^2 \gamma^2 + \beta^2} \right\} \quad (45)$$

$$\text{and } N_m = \int_0^1 \cdots \int_0^1 dz_1 \cdots dz_m \Lambda_m^{-1} \exp \left[-r^2 / z^2 \Lambda_m \right] \quad (45b)$$

$$\text{where } \Lambda_m = \alpha^{-2} \sum_{i=1}^m z_i^2 + \beta^{-2} + z^{-2} \gamma^{-2}. \quad (45c)$$

We may note in particular that N_1 may be evaluated analytically in terms of the error function. The resulting expression is quite complicated, except in the case where β and γ go to infinity, in which case we get

$$N_1(z, r) = \alpha^2 \sqrt{\pi} [1 - \text{erf}(g)] / 2g \quad (46a)$$

$$\text{where } g = r \alpha / z. \quad (46b)$$

Turning our attention to the power received, we obtain an expression similar to Eqs. (44) and (45), viz.

$$P(z, R) = P_0 e^{-\tau} \sum_{m=0}^{\infty} \frac{\tau^m}{m!} P_m(z, R) \quad (47)$$

$$\text{where } P_0 = 1 - \exp \left\{ - \frac{R^2 \beta^2 \gamma^2}{z^2 \gamma^2 + \beta^2} \right\} \quad (48a)$$

$$\text{and } P_m = \int_0^1 \cdots \int_0^1 dz_1 \cdots dz_m \{1 - \exp [-R^2/z^2 \Lambda_m]\} . \quad (48b)$$

Note that, from Eqs. (47) and (48), Eq. (13) may be obtained trivially.

As with N_1 , P_1 is also analytic, and in the simple case of $\beta, \gamma \rightarrow \infty$, we obtain

$$P_1(z, R) = 1 - e^{-G^2} + G \sqrt{\pi} [1 - \text{erf}(G)] . \quad (49)$$

The number of terms required for the convergence of the series in Eqs. (44) and (47) grows steadily with τ_g and so in some cases for large optical thicknesses it may become prohibitively expensive to use it. Nevertheless, these results have one use in that Tam and Zardecki (Ref. 13) have shown that the m th order terms in Eqs. (44) and (47) correspond to the contribution from m th order scattering. This in itself is a useful result.

Another use suggests itself, however. The Gaussian phase function is simply a model, with the parameters α and ω_0 available for adjustment to match "real"

scattering patterns. Since we now have a simple expression for the singly scattered contribution, we may compare it with that produced by a real phase function and adjust α and ω_0 accordingly. Then we may use the results outlined above to estimate the multiply scattered contribution from such a phase function. Comparisons with second and higher order contributions are also possible. This should increase our confidence in the worth of results obtained from a model phase function.

7.0 NUMERICAL RESULTS

In this section, we shall present some typical results based on our exact formulation and the Arnush-Stotts type approximate method from selected computational results. We shall examine 4 phase function models: Gaussian (Eq. 17), both exponential models (Eqs. 25a and 27a), and the binomial model with $\mu = 3/2$ (Eq. 29a). To simplify discussion, we shall refer to the phase function model of Eq. (25) as the exponential model, and that of Eq. (27) as the sea-water model.

We start by examining the phase functions themselves. It is, of course, much simpler to plot the normalized phase function, \bar{P} , rather than P , where \bar{P} is defined by

$$\bar{P} = 2\pi P / \alpha^2. \quad (50)$$

Unlike P , \bar{P} is now a function of only one variable, $\alpha\psi$. In Figure 1, we plot \bar{P} against $\alpha\psi$, for $0 \leq \alpha\psi \leq 3.5$.

The next function we examine graphically is Ω_0 . In Fig. 2 we plot Ω_0/τ_s against y for all 4 phase functions, as well as the Arnush-Stotts approximation. From this log-log plot, the asymptotic behavior of the 4 phase functions is apparent, particularly that of the sea-water phase function, which has no asymptotic expansion. Although it is not obvious from this figure, the curve for the binomial phase function actually lies slightly above the sea-water curve for y values less than about 3. Finally, we note that for y values greater than 2, the results obtained by the Arnush-Stotts approximation differ markedly from those by our exact formulation, rapidly approaching large negative values for y greater than 4.

We now turn to a discussion of the amplification factor, A , and the power received, P , as predicted by these 4 models, and also the Arnush-Stotts approximation. We have evaluated both A and P for G between 0.01 and 1.0, and τ_s between 0.5 and 15.0. (Throughout, we have assumed $\beta, \gamma \rightarrow \infty$.)

In the Appendix to this report, we have included a listing of the FORTRAN program used to generate this data, along with a brief explanation and sample output.

In Figure 3 we plot A against G for a series of τ_s values, for the Gaussian model. In Figure 4 we plot P against τ_s (assuming unit incident power) for a series of values of G , again for the Gaussian model. Also shown on this plot is the transmission, T , which represents the power that would be received if all scattered light was lost. These two graphs clearly indicate the important role that forward scattering can play in the detection of transmitted beams, especially for optical thicknesses of the order of 10 or higher.

In Figure 5 we plot A as a function of G for $\tau_s = 4.0$ and 10.0 , for all 4 model phase functions, for our formulation as well as the Arnush-Stotts approximation. In the latter case, one finds values of A which are less than unity. Note that the binomial and sea water curves cross for both τ_s values (cf. Fig. 2). For large values of G , we see that there is little to choose between the four phase function models.

In Figure 6, we plot $(A - 1.0)$ against G in log-log form for $\tau = 1.0$, in order to emphasize the linear relationship implied by Eq. (A3) in the Appendix. We see that for G less than 0.3 , the integral term in Eq. (A3) makes a negligible contribution. In Figure 7, we plot $(A - 1.0)$ against G for $\tau = 5.0$. Here we see that the integral term in Eq. (A3) is starting to make a contribution. Also in this graph we have included the Arnush-Stotts and Dolin-Fante approximation results. The Dolin-Fante result was not included in Figure 6 as it could not be distinguished from the exact result for the Gaussian phase function.

8.0 CONCLUDING REMARKS

The propagation of a laser beam in an optically dense medium such as a fog, dust storm, or smoke is a problem of growing importance, both for communication and detection purposes. Although such dense media lead to a significant attenuation of the primary beam, much of the scattered radiation may still be found close to the beam axis and will, thus, be available for detection by a suitable detector.

In this report, we have examined the spreading of a laser beam using the small-angle scattering approximation to the equation of transfer. This approximation appears eminently suited for the study of beam propagation in fog, dust, or smoke media, where the scattering phase function is highly

anisotropic. As well as the standard Gaussian model phase function, three other model phase functions have also been examined. The Gaussian functional form was used to describe the initial beam spread and profile, although the analysis is somewhat simpler (and the resulting expressions tidier) if the limiting case is taken.

All the numerical results presented in this paper have been based on the assumption of a narrow, collimated beam. We may remark, however, that the results and expressions presented in this report (e.g., Eq. 20) may be applied with full generality.

We have also examined a number of approximations which have been used to further simplify the expressions we have derived. The Arnush-Stotts approximation is quite suitable for use in the asymptotic regions, at large distances from the beam axis. However, the behavior of the solutions close to the beam axis is governed by the parameter \tilde{q} , the zeroth moment of the phase function. This moment weights the contribution from scattering through very small angles far more highly than does the parameter $\overline{\psi^2}$, the rms scattering angle, or third moment. In fact, for small R (i.e., small G), one may expand Eq. (23) to first order (cf. Eq. A3)

$$A(\tau_s, G) \approx 1 + \tau_s \tilde{q} G + \dots \quad (51)$$

Finally, we may remark that the method proposed by Tam and Zardecki makes a useful contribution by providing a connection between real (Mie) phase functions, and the parameters which must be used in the model phase functions used in this report. Further work on the applications of this method is recommended.

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APPENDIX: COMPUTATIONAL DETAILS

The numerical results presented in Figs. 3 to 7 were obtained from a relatively simple computer program, consisting of less than 100 executable statements. A full listing, and partial output, are included in this appendix.

The task of this program is to evaluate the numerical integrals in Eqs. (23), (25d), (27d) and (30'), for a series of values of τ_s between 0.5 and 15.0, and a series of values of G up to 1.0. For comparison, the Arnush-Stotts approximation, Eq. (39), is also computed. We have included the full results for τ_s values of 4.0 and 10.0, which may be read in conjunction with Fig. 5.

Although infinite integrals of this type are often handled by Gauss-Laguerre quadrature, this method was found wanting, due to the oscillatory nature of the integrand. Instead we have employed Simpson's rule, up to a finite cut off, allowing for the remainder of the integral by the following result:

If

$$f(\chi) \approx \phi(\chi) \quad \text{for } \chi \geq X$$

and

$$\int_0^\infty \phi(\chi) d\chi = \Phi \quad \text{is known,}$$

then

$$\int_0^\infty f(\chi) d\chi \approx \Phi + \int_0^X [f(\chi) - \phi(\chi)] d\chi. \quad (A1)$$

To apply this result, we note Eq. (41), and choose X such that

$$e^{\Omega_0} \approx 1 + \tau_s \tilde{q} G/X, \quad X \geq X. \quad (A2)$$

Thus

$$\int_0^\infty J_1(\chi) e^{\Omega_0} d\chi \approx 1 + \tau_s \tilde{q} G + \int_0^X J_1(\chi) (e^{\Omega_0} - 1 - \tau_s \tilde{q} G/X) d\chi. \quad (A3)$$

Due to the wide range of values of G which we have used (three orders of magnitude) it is necessary to vary the step size accordingly. Thus we have used a step size equal to G , up to $G = 0.22$. A step size larger than this is unwise, due to the variation in the J_1 term. Thus, when we reach $G = 0.22$, a larger set of J_1 values is computed and stored, to be used for the remaining values of G .

As pointed out above Eq. (41), \tilde{q} is undefined for the sea water phase function, so we have set $\tilde{q} = 0$ in this case. As a result, we are forced to choose a considerably higher value of X in order to satisfy Eq. (A2). In fact, if the sea water phase function was dropped from consideration, the time (and cost) of these calculations would be cut at least in half. As it is, the results we have obtained for this phase function must be considered distinctly less accurate than the others.

```

1      PROGRAM PROPGTN(OUTPUT,TAPE6=OUTPUT)
      C
      C      THIS PROGRAM COMPUTES THE SIGNAL DETECTED BY A COAXIAL
      C      DISK DETECTOR WHEN A LASER BEAM HAS TRAVERSED A MEDIUM
5      C      WITH A HIGHLY FORWARD-PEAKED PHASE FUNCTION, USING THE
      C      EQUATION OF TRANSFER IN THE SMALL ANGLE APPROXIMATION.
      C
      C      4 DIFFERENT SCATTERING PHASE FUNCTIONS ARE CONSIDERED:
      C
10     C      1) GAUSSIAN      2*ALPHA**2*EXP(-(ALPHA*PSI)**2)
      C
      C      2) EXPONENTIAL    ALPHA**2*EXP(-ALPHA*PSI)
      C
      C      3) SEA WATER      ALPHA*EXP(-ALPHA*PSI)/PSI
15     C
      C      4) BINOMIAL      3*ALPHA**2*(1+(ALPHA*PSI)**2)**(-5/2)
      C
      C      THE ARNUSH-STOTTS APPROXIMATION IS ALSO COMPUTED FOR
      C      THESE PHASE FUNCTIONS. (NOTE THAT THIS APPROXIMATION
20     C      GIVES IDENTICAL RESULTS FOR PHASE FUNCTIONS 3 AND 4)
      C
      C      DEFINITIONS.....
      C      SP      = SORT(PI)
      C      X      = INTEGRATION VARIABLE
25     C      DX      = INTEGRATION STEP SIZE
      C      Y      = X / G
      C      G      = RADIUS OF DETECTOR * ALPHA / PATH LENGTH (Z)
      C      TRANS   = TRANSMITTANCE
      C      OMEGA   = INTEGRAL FROM 0 TO 2 OF THE FOURIER
30     C      TRANSFORM OF THE PHASE FUNCTION
      C      AMP     = AMPLIFICATION FACTOR
      C      SA      = ARNUSH-STOTTS APPROXIMATION FOR AMP
      C      SIGNAL  = DETECTED POWER FOR UNIT TRANSMITTED POWER
      C      AS      = ARNUSH-STOTTS APPROXIMATION FOR SIGNAL
35     C      Q      = PARAMETER IN THE EXPANSION OF OMEGA:
      C      OMEGA = TAU ( 1 - Y*Y / Q + ..... )
      C      P      = Q * G * G / 4 * TAU
      C
      C      EXTERNAL SUBROUTINES (FROM FTNHLIB)
40     C      BJ0R   BESSEL FUNCTION OF ORDER 0
      C      BJ1R   BESSEL FUNCTION OF ORDER 1
      C      ERF     ERROR FUNCTION

```

```

C
45  DIMENSION BB(2560),SIGNAL(4),AMP(4),AS(3),SA(3),Q(3),GG(20)
      DATA SP,DX / 1.77245385,0.05 / ,Q / 12.0,2.0,6.0 /
      DATA GG / 0.01,0.025,0.05,0.075,0.1,0.125,0.15,0.175,0.2,
1      0.25,0.3,0.35,0.4,0.45,0.5,0.6,0.7,0.8,0.9,1.0 /
      X=-DX/2.0

C
50  SET UP BB ARRAY OF J1 BESSEL FUNCTIONS
C
      DO 5 I=1,2560
      X=X+DX
      CALL BJ1R(X,BB(I),IER)
55  5 CONTINUE

C
      ASSUME INTEGRAL FROM 128 TO INFINITY = JO(128.)
C
      CALL BJOR(128.0,TAIL,IER)
60  WRITE(6,10) TAIL
10  FORMAT(50X,'TAIL =',F10.5)

C
      DO LOOP OVER OPTICAL THICKNESS, TAU
C
65  DO 70 J=1,16
      TAU=J-1
      IF(J.EQ.1) TAU=0.5
      TRANS=EXP(-TAU)
      WRITE(6,15) TAU,TRANS
70  15 FORMAT(1H1,30X,'OPTICAL THICKNESS, TAU =',F5.1,10X,'TRANSMIT*
1      *TANCE =',E12.4, '//,30X,'GAUSSIAN*,20X,'EXPONENTIAL*,14X,
2      *SEA WATER*,14X,'BINOMIAL*,//,15X,'G*,9X,'EXACT*,7X,
3      *APPROX*,2(11X,'EXACT*,7X,'APPROX*),8X,'EXACT*,//)

C
75  DO LOOP OVER GEOMETRY FACTOR, G
C
      DO 50 L=1,20
      G=GG(L)
      AMP(1)=AMP(2)=AMP(3)=AMP(4)=TAIL
80  X=-DX/2.0

C
      INTEGRAL FROM 0.0 TO 128.0 : 2560 EQUAL STEPS
C
      DO 20 I=1,2560

```

```

85      X=X+DX
        B1=BB(I)
        Y=X/G
        CALL EFF(0.5*Y,ERFY)
        OMEGA=TAU*SP*ERFY/Y
90      AMP(1)=AMP(1)+B1*EXP(OMEGA)*DX
        OMEGA=TAU/SORT(1.0+Y**2)
        AMP(2)=AMP(2)+B1*EXP(OMEGA)*DX
        OMEGA=TAU/Y*ALOG(Y*SORT(1.0+Y**2))
        AMP(3)=AMP(3)+B1*EXP(OMEGA)*DX
95      OMEGA=2.0*TAU/Y
        IF(Y.LT.20) OMEGA=OMEGA-TAU*EXP(-Y)*(1.0+2.0/Y)
        AMP(4)=AMP(4)+B1*EXP(OMEGA)*DX
20      CONTINUE

C
C      NOW CALCULATE THE ARNUSH-STOTTS APPROXIMATION
C      (RESULTS FOR PHASE FUNCTIONS 3 & 4 ARE IDENTICAL)
C
        DO 25 I=1,3
        P=Q(I)*G*G/4.0/TAU
105      AS(I)=1.0-EXP(-P)
        SA(I)=AS(I)/TRANS
        SIGNAL(I)=AMP(I)*TRANS
25      CONTINUE
        SIGNAL(4)=AMP(4)*TRANS
110      WRITE(6,30) G,(AMP(I),SA(I),I=1,3),AMP(4)
30      FORMAT(11X,F7.3,3(3X,2G13.5),G13.5)
        WRITE(6,35) (SIGNAL(I),AS(I),I=1,3),SIGNAL(4)
35      FORMAT(18X,3(3X,2G13.5),G13.5)
50      CONTINUE
115      WRITE(6,60)
60      FORMAT(/,10X,*THE FIRST LINE IS THE AMPLIFICATION FACTOR, THE *,
1        *SECOND LINE IS THE RECEIVED POWER (FOR UNIT INCIDENT POWER)*
70      CONTINUE
        STOP
120      END

```

OPTICAL THICKNESS, $\tau = 4.0$

TRANSMITTANCE = .1832E-01

	GAUSSIAN		EXPONENTIAL		SEA WATER		BINOMIAL
G	EXACT	APPROX	EXACT	APPROX	EXACT	APPROX	EXACT
.010	1.0822	.40947E-02	1.0427	.68247E-03	1.2395	.20474E-02	1.0926
	.19822E-01	.74997E-04	.19098E-01	.12500E-04	.22702E-01	.37499E-04	.20011E-01
.025	1.2377	.25587E-01	1.1177	.42653E-02	1.5756	.12795E-01	1.2661
	.22669E-01	.46864E-03	.20471E-01	.78122E-04	.28858E-01	.23435E-03	.23189E-01
.050	1.5805	.10228	1.2612	.17059E-01	2.1599	.51162E-01	1.6239
	.23582E-01	.18732E-02	.23100E-01	.31245E-03	.39560E-01	.93706E-03	.29744E-01
.075	1.9597	.22985	1.4212	.38376E-01	2.7831	.11505	2.0590
	.35894E-01	.42099E-02	.26067E-01	.70288E-03	.50975E-01	.21072E-02	.37712E-01
.100	2.4258	.40795	1.6038	.68205E-01	3.4400	.20436	2.5600
	.44431E-01	.74719E-02	.29374E-01	.12492E-02	.63006E-01	.37430E-02	.46088E-01
.125	2.9520	.63609	1.8014	.10653	4.1246	.31898	3.1183
	.54069E-01	.11650E-01	.32995E-01	.19512E-02	.75544E-01	.58422E-02	.57114E-01
.150	3.5326	.91361	2.0147	.15334	4.6316	.45873	3.7267
	.64702E-01	.16733E-01	.36901E-01	.28085E-02	.88494E-01	.84020E-02	.68256E-01
.175	4.1625	1.2398	2.2424	.20861	5.5567	.62344	4.3788
	.76239E-01	.22707E-01	.41071E-01	.38208E-02	.10178	.11419E-01	.80200E-01
.200	4.8371	1.6136	2.4835	.27231	6.2963	.81286	5.0691
	.88594E-01	.29554E-01	.45487E-01	.49875E-02	.11532	.14888E-01	.92844E-01
.250	6.3035	2.5002	3.0023	.42489	7.8064	1.2648	6.5447
	.11545	.45793E-01	.54989E-01	.77821E-02	.14298	.23165E-01	.11987
.300	7.9011	3.5637	3.5649	.61079	9.3416	1.8119	8.1186
	.14471	.65272E-01	.65293E-01	.11187E-01	.17110	.33187E-01	.14870
.350	9.6024	4.7927	4.1659	.82967	10.886	2.4514	9.7613
	.17587	.87781E-01	.76302E-01	.15196E-01	.19939	.44898E-01	.17878
.400	11.383	6.1739	4.8008	1.0811	12.429	3.1795	11.448
	.20848	.11308	.87930E-01	.19801E-01	.22764	.58235E-01	.20968
.450	13.220	7.6931	5.4655	1.3647	13.959	3.9925	13.159
	.24214	.14090	.10010	.24395E-01	.25567	.73126E-01	.24101
.500	15.094	9.3347	6.1562	1.6798	15.470	4.8860	14.875
	.27647	.17097	.11275	.30767E-01	.28335	.89490E-01	.27245
.600	18.885	12.919	7.6027	2.4025	18.414	6.8949	18.272
	.34589	.23662	.13925	.44003E-01	.33727	.12628	.33467
.700	22.635	16.791	9.1171	3.2438	21.228	9.1646	21.554
	.41457	.30754	.16699	.59412E-01	.38880	.16786	.39477
.800	26.253	20.014	10.680	4.1977	23.892	11.450	24.664
	.48584	.38122	.19561	.76984E-01	.43760	.21337	.45173
.900	29.673	24.858	12.274	5.2574	26.398	14.302	27.570
	.54349	.45529	.22481	.96293E-01	.48350	.26195	.50497
1.000	32.651	28.308	13.886	6.4155	28.742	17.073	30.258
	.60169	.52763	.25433	.11750	.52643	.31271	.55419

THE FIRST LINE IS THE AMPLIFICATION FACTOR, THE SECOND LINE IS THE RECEIVED POWER (FOR UNIT INCIDENT POWER)

OPTICAL THICKNESS, $\tau = 10.0$

TRANSMITTANCE = .4540E-04

	GAUSSIAN		EXPONENTIAL		SEA WATER		BINDMIAL
G	EXACT	APPROX	EXACT	APPROX	EXACT	APPROX	EXACT
.010	1.6482	.66078	1.1333	.11013	2.2419	.33039	1.6767
	.74828E-04	.30000E-04	.51453E-04	.50000E-05	.10178E-03	.15000E-04	.76123E-C4
.025	8.6159	4.1296	2.0765	.68832	8.5240	2.0649	7.4024
	.39116E-03	.18748E-03	.94274E-04	.31250E-04	.38699E-03	.93746E-04	.33607E-C3
.050	25.201	16.514	7.1070	2.7531	25.702	8.2584	22.560
	.11441E-02	.74972E-03	.32266E-03	.12499E-03	.11669E-02	.37493E-03	.10242E-C2
.075	52.616	37.138	13.436	6.1941	51.115	18.577	46.385
	.23888E-02	.16861E-02	.60998E-03	.28121E-03	.23206E-02	.84339E-03	.21059E-C2
.100	90.753	65.980	21.204	11.010	85.476	33.015	74.395
	.41202E-02	.29955E-02	.96266E-03	.49988E-03	.38806E-02	.14989E-02	.36045E-C2
.125	139.38	103.01	31.087	17.201	128.42	51.564	121.37
	.63278E-02	.46765E-02	.14113E-02	.78094E-03	.58301E-02	.23410E-02	.55103E-C2
.150	198.33	148.18	43.167	24.766	179.57	74.214	172.12
	.90040E-02	.67273E-02	.19598E-02	.11244E-02	.81526E-02	.33693E-02	.73142E-C2
.175	267.43	201.44	57.393	33.702	238.63	100.95	231.44
	.12141E-01	.91454E-02	.26056E-02	.15301E-02	.10834E-01	.40832E-02	.10508E-C1
.200	346.50	262.74	73.724	44.009	305.29	131.76	299.15
	.15731E-01	.11928E-01	.33471E-02	.19980E-02	.13860E-01	.59820E-02	.13581E-C1
.250	533.78	409.15	112.59	68.725	460.27	205.53	453.86
	.24234E-01	.18575E-01	.51116E-02	.31201E-02	.20896E-01	.91312E-02	.20832E-C1
.300	758.49	586.76	159.59	98.896	642.28	290.36	649.53
	.34435E-01	.26639E-01	.72454E-02	.44899E-02	.29160E-01	.13409E-01	.29488E-C1
.350	1018.7	794.78	214.56	134.50	849.20	401.04	869.27
	.46251E-01	.36083E-01	.97409E-02	.61063E-02	.38553E-01	.18207E-01	.39465E-C1
.400	1312.5	1032.3	277.33	175.51	1078.9	522.34	1116.1
	.59589E-01	.46866E-01	.12591E-01	.79681E-02	.48983E-01	.23714E-01	.50172E-C1
.450	1637.7	1298.3	347.75	221.89	1329.5	658.99	1338.1
	.74350E-01	.58942E-01	.15788E-01	.10074E-01	.60359E-01	.29918E-01	.63020E-C1
.500	1991.9	1591.6	425.63	273.62	1599.0	810.70	1683.2
	.90432E-01	.72257E-01	.19323E-01	.12422E-01	.72594E-01	.36806E-01	.76416E-C1
.600	2778.2	2254.9	603.05	392.93	2187.4	1157.9	2334.3
	.12613	.10237	.27378E-01	.17839E-01	.99310E-01	.52568E-01	.10598
.700	3652.2	3011.2	808.09	533.09	2630.6	1560.9	3053.2
	.16581	.13671	.36667E-01	.24202E-01	.12851	.70864E-01	.13361
.800	4594.5	3847.9	1039.2	693.69	3515.9	2016.2	3824.3
	.20859	.17469	.47170E-01	.31493E-01	.15962	.91536E-01	.17252
.900	5586.3	4751.7	1294.6	874.25	4232.4	2520.0	4633.1
	.25362	.21573	.58777E-01	.39691E-01	.19215	.11441	.21234
1.000	6609.6	5706.9	1572.8	1074.2	4970.1	3068.1	5455.4
	.30007	.25918	.71407E-01	.48771E-01	.22564	.13929	.24318

THE FIRST LINE IS THE AMPLIFICATION FACTOR, THE SECOND LINE IS THE RECEIVED POWER (FOR UNIT INCIDENT POWER)

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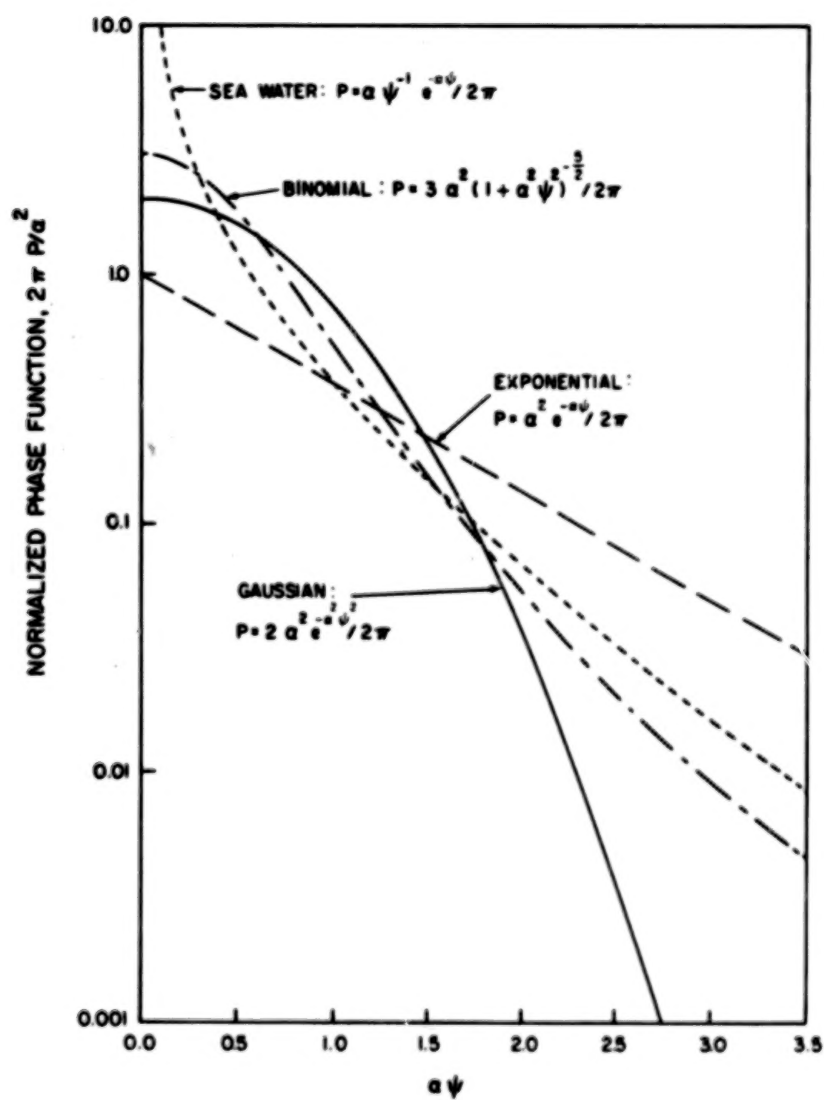


FIGURE 1. Normalized phase function \bar{P} vs $\alpha\psi$ for four model phase functions.

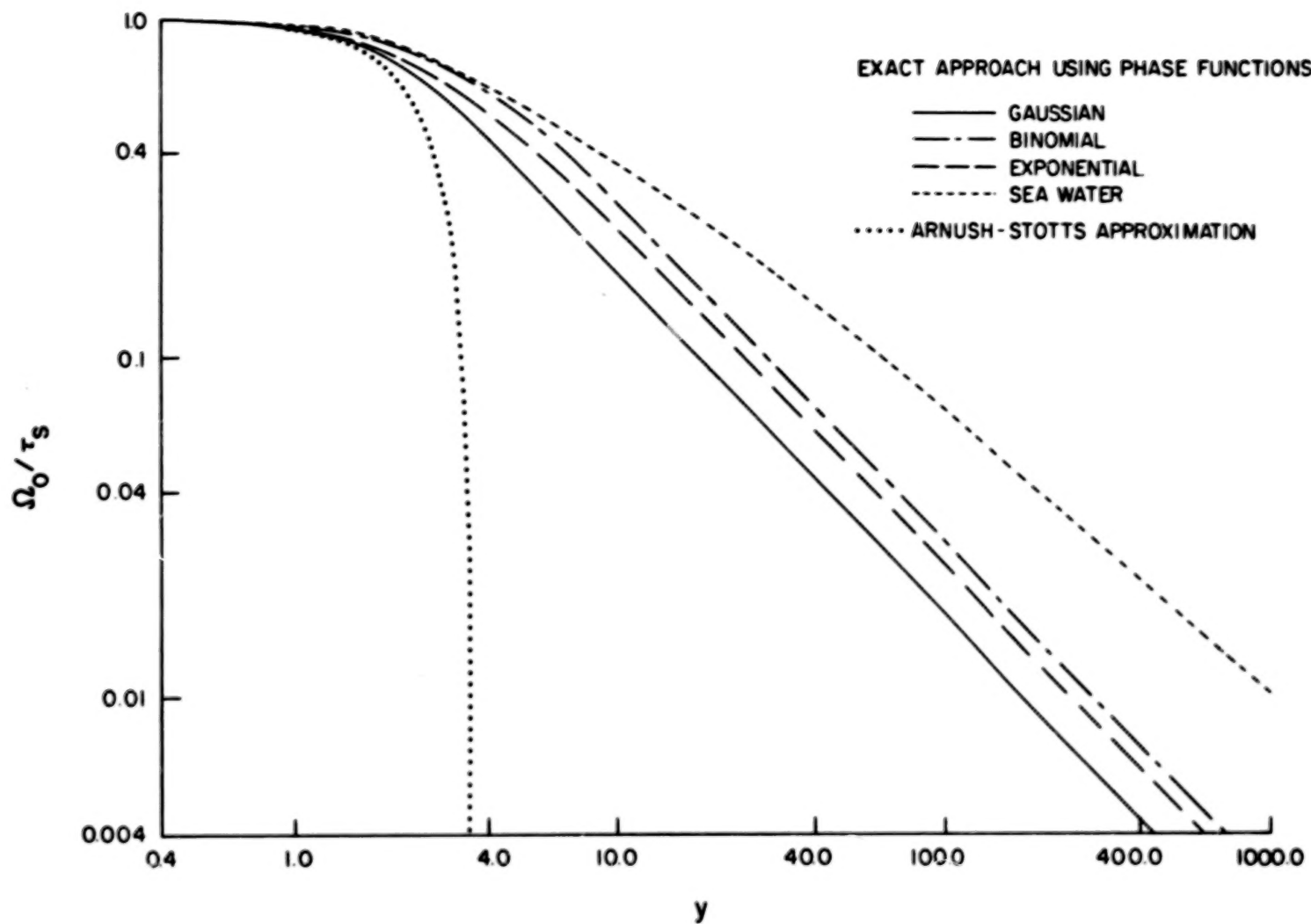


FIGURE 2. Ω_0/τ_s vs. y for four model phase functions and for the Arnush-Stotts approximation.

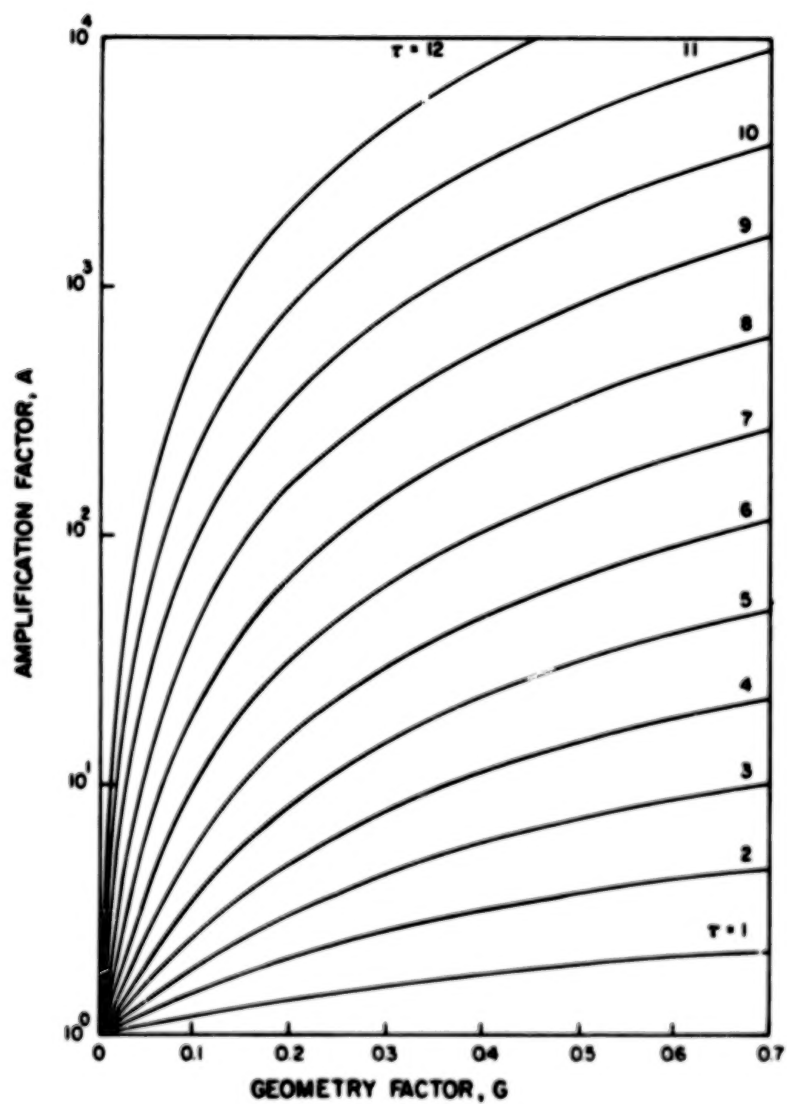


FIGURE 3. Amplification factor A vs. geometry factor G for the Gaussian phase function.

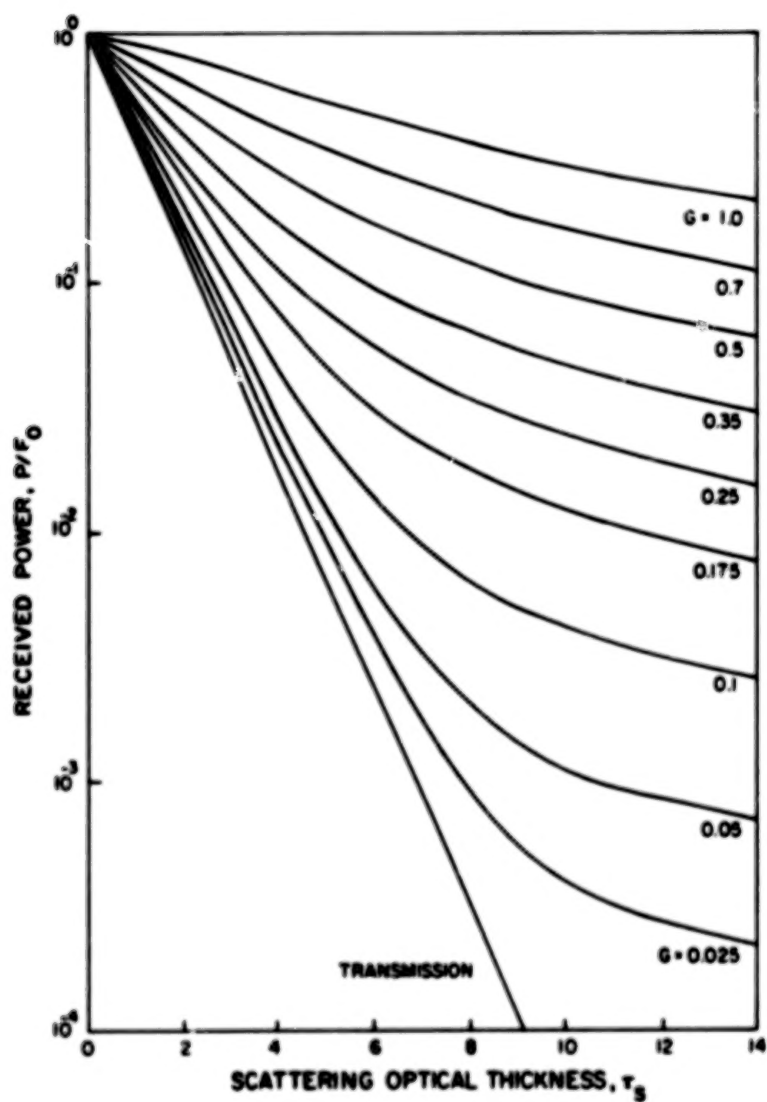


FIGURE 4. Normalized power received vs. scattering optical thickness for the Gaussian phase function.

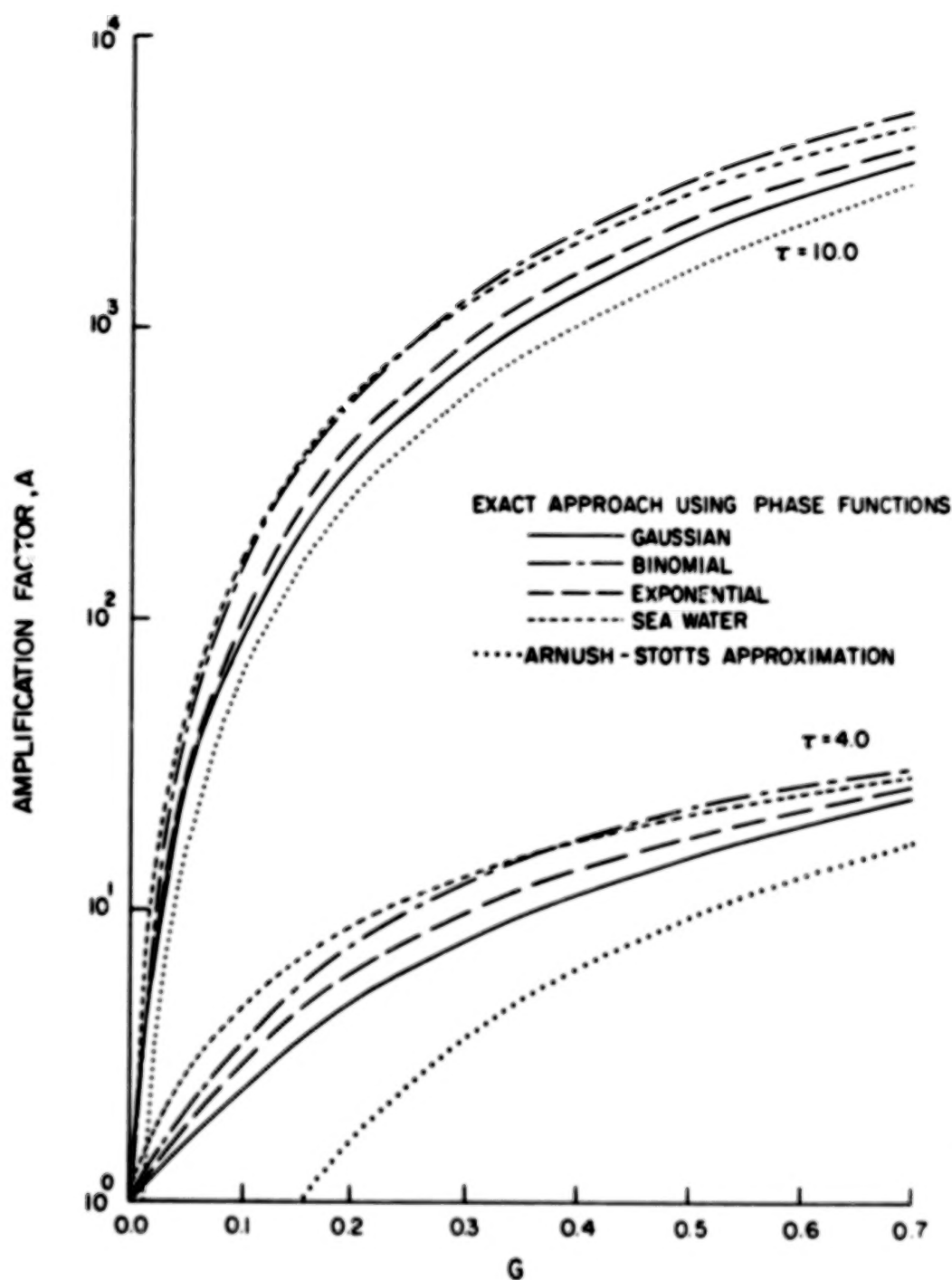


FIGURE 5. Amplification factor A vs. geometry factor G for two values of τ_s and four model phase functions for our approach and the Arnush-Stotts approximation.

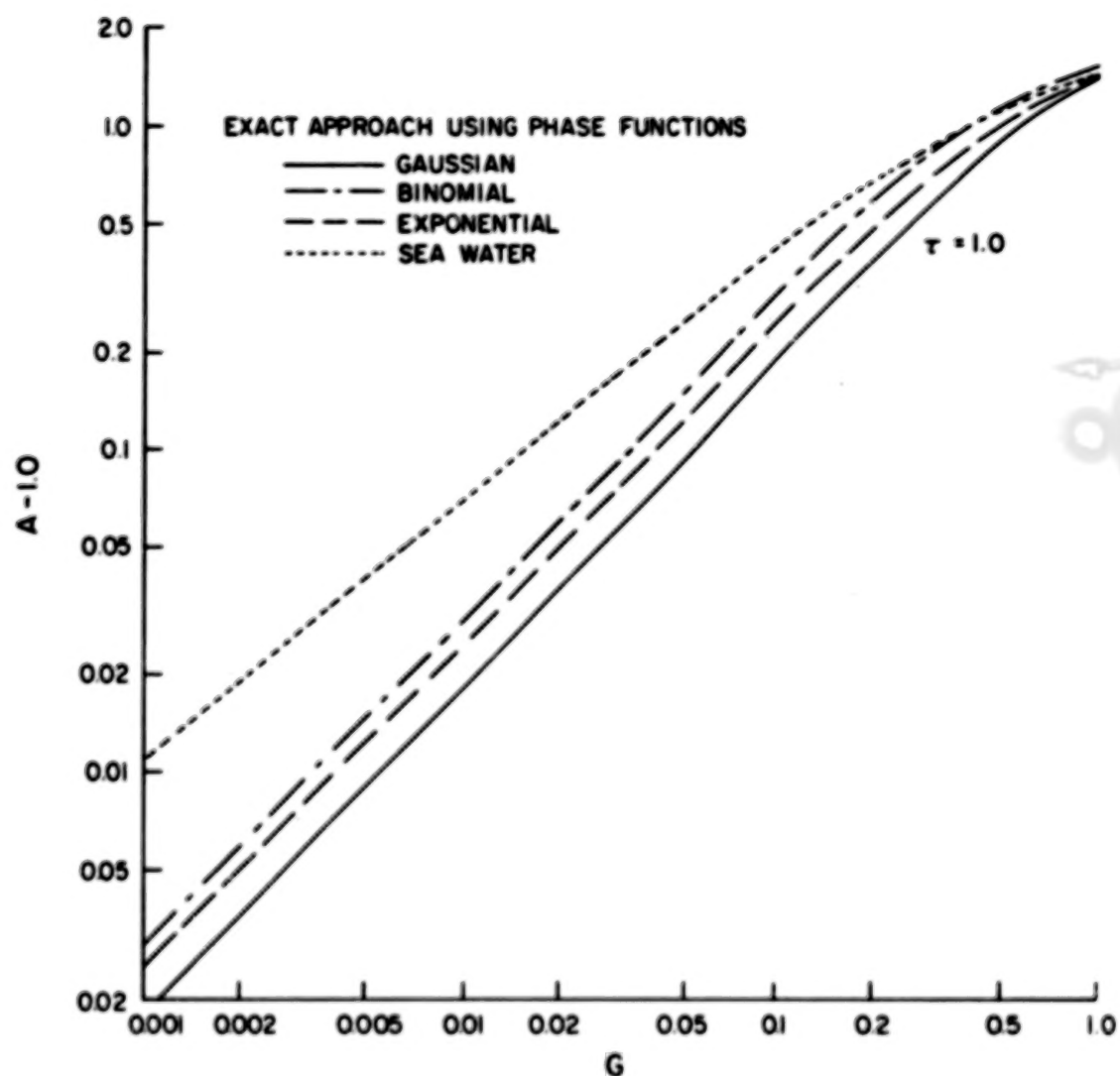


FIGURE 6. Subtracted amplification factor ($A - 1.0$) vs. geometry factor (G) for $\tau = 1.0$.

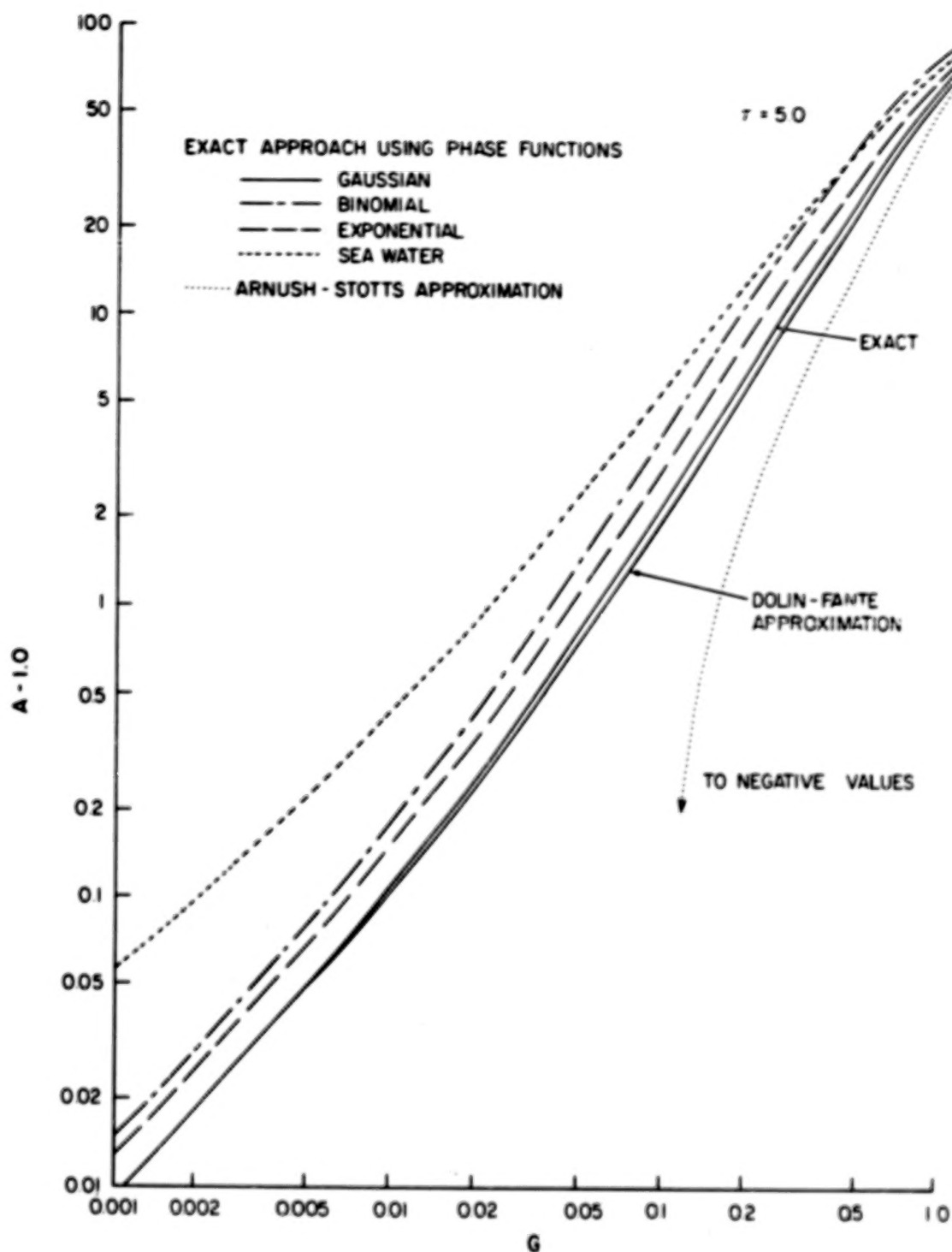


FIGURE 7. Subtracted amplification factor ($A - 1.0$) vs. geometry factor (G) for $\tau = 5.0$.

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16. ABSTRACT This report examines the broadening, and the signal power detected, of a laser beam traversing an anisotropic scattering medium using the small-angle approximation to the radiative transfer equation in which photons suffering large-angle deflections are neglected. To obtain tractable answers, simple Gaussian and non-Gaussian functions for the scattering phase functions are assumed. In addition, descriptions of two other approximate approaches which have been employed in the field to further simplify the small-angle approximation solutions are given, and the results obtained by one of them are compared with those obtained with our approach. Also described is an exact method for obtaining the contribution of each higher order scattering to the radiance field, but no results will be presented here.					
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